Turbulent flow in Lake Kivu

K.D. Anderson, H. Diedericks, A. Dlamini, S. Fourie,
N. Freeman, M. Khalique, O.A.I. Noreldin, D.P. Mason,
N. Nkomo, T. Pramjeeth, T. Sekgobela

Industry Representative: D. Ndanguza

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Introduction

Killer lakes

- Limnic eruptions every $\approx$ 750-1000 years, causes unknown
- Lake Monoun, Cameroon - 37 people killed
- Lake Nyos, Cameroon - 1746 people, 3500 livestock killed
- Lake Kivu, DRC & Rwanda - ?

Problem: Lake Kivu approximately 2 000 times larger than Lake Monoun...

Lake Kivu (cont.)

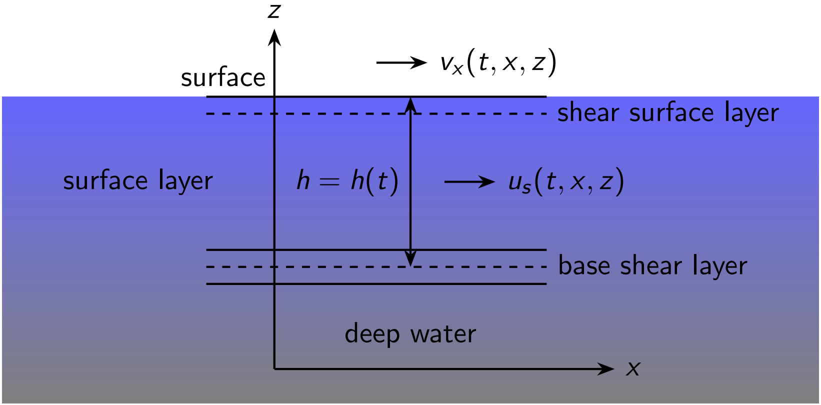
- Limnic eruption = catastrophy
- Threatens biozone upon which communities depend
- Not the only threat to the biozone...
- Wind stress, radiation and heat input on the surface create mean current and turbulence
- Leads to mixing in vertical direction and potential changes in concentrations of biozone layer

Problem Solving Approach

- One subgroup to work on existing model (Imberger)
- One subgroup to work on the ODEs of the existing model
- One subgroup to collect data on Lake Kivu which could be used by whole group

Modelling

Turbulent flow in biozone



- Surface Layer Hydraulics
- Chapter 6 in Environmental Fluid Mechanics by J. Imburger

Continuity equation

- Continuity equation

$$\underline{\nabla} \cdot \underline{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_z}{\partial z} = 0 \quad (1)$$

- Substitute $v_i = \bar{v}_i + v_i'$ into (1)

$$\frac{\partial}{\partial x}(\bar{v}_x + v_x') + \frac{\partial}{\partial z}(\bar{v}_z + v_z') = 0$$

- Simplify and take the mean ($\overline{\overline{v}} = \bar{v}$, $\overline{v'} = 0$)

$$\frac{\partial \bar{v}_x}{\partial x} + \frac{\partial \bar{v}_z}{\partial z} + \frac{\partial v_x'}{\partial x} + \frac{\partial v_z'}{\partial z} = 0$$

- Mean velocity satisfies (1), velocity fluctuations satsifies (1)

- Heat equation

$$\rho c_v \frac{D\theta}{Dt} = \rho c_v \left[\frac{\partial \theta}{\partial t} + \left(v_x \frac{\partial \theta}{\partial x} + v_z \frac{\partial \theta}{\partial z}\right)\right] = -k\left(\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial z^2}\right) \quad (2)$$

- Substitute

$$\theta = \bar{\theta} + \theta', \quad v_i = \bar{v}_i + v_i', \quad i = \{x, z\}$$

into (2)...

- After lots of algebra and taking the mean

$$\frac{\partial \bar{\theta}}{\partial t} + \bar{v}_x \frac{\partial \bar{\theta}}{\partial x} + \bar{v}_z \frac{\partial \bar{\theta}}{\partial z} = -\frac{\partial}{\partial x}\left(\overline{v_x'\theta'}\right) - \frac{\partial}{\partial z}\left(\overline{v_z'\theta'}\right) - \frac{k}{\rho_0 c_v}\left(\frac{\partial^2 \bar{\theta}}{\partial x^2} + \frac{\partial^2 \bar{\theta}}{\partial z^2}\right)$$

- Mean energy equation

$$\frac{\partial \bar{\theta}}{\partial t} = -\frac{\partial}{\partial z}\left(\overline{v_z'\theta'}\right) + \frac{1}{\rho_0 c_v}\frac{\partial q}{\partial z}, \quad q = -k\frac{\partial \theta}{\partial z} \quad (3)$$

Mean density field

- Simplified equation of state

$$\alpha = -\frac{1}{\rho}\frac{\partial \rho}{\partial \theta}\bigg|_{\theta=\theta_0} \qquad \rho = -\alpha\rho_0\theta \tag{4}$$

- Substitute $\rho = \bar{\rho} + \rho'$ into (4)

$$\bar{\rho} = -\alpha\rho_0\bar{\theta}, \quad \rho' = -\alpha\rho_0\theta' \tag{5}$$

- Substitute (5) into (3) and simplify (lots of algebra again)
- Mean density field equation

$$\frac{\partial \bar{\rho}}{\partial t} = -\frac{\partial\left(\overline{\rho' v_z'}\right)}{\partial z} - \frac{\alpha}{c_v}\frac{\partial \bar{\theta}}{\partial z} \tag{6}$$

Mean momentum equation

- x component of Navier-Stokes

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_z \frac{\partial v_z}{\partial z} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_z}{\partial z^2} \right) \tag{7}$$

- Substitute $v_x = \bar{v}_x + v_x'$, $p = \bar{p} + p'$ into (7), simplify, take the mean
- Mean momentum equation

$$\frac{\partial \bar{v}_x}{\partial t} = -\frac{\partial}{\partial z} \left(\overline{v_x' v_z'} \right) \tag{8}$$

Turbulent kinetic energy

- Navier-Stokes with the Boussinesq approximation (tensor notation)

$$\rho_0 \left(\frac{\partial v_i}{\partial t} + v_k v_{i,k} \right) = -p_{,i} + \mu v_{i,kk} - \rho g \delta_{zi} \tag{9}$$

- Substitute $v_i = \bar{v}_i + v_i'$, $p = \bar{p} + p'$, $\rho = \bar{\rho} + \rho'$ and expanding and taking the mean

$$\rho_0 \left(\frac{\partial \bar{v}_i}{\partial t} + \bar{v}_k \bar{v}_{i,k} + \overline{v_k' v_{i,k}'} \right) = -\bar{p}_{,i} + \mu \bar{v}_{i,kk} - \bar{\rho} g \delta_{zi}$$

Turbulent kinetic energy

- Mean momentum equation

$$\rho_0 \frac{\bar{\mathrm{D}}\bar{v}_i}{\mathrm{D}t} = -\bar{p}_{,i} + \mu \bar{v}_{i,kk} - \rho_0 \overline{\left(v_i' v_k'\right)}_{,k}, \qquad \frac{\bar{\mathrm{D}}}{\mathrm{D}t} = \frac{\partial}{\partial t} + (\bar{v} \cdot \underline{\nabla})\bar{v}_i \tag{10}$$

- Taking scalar product with $\bar{v}_i$ and doing some algebra
- Mean kinetic energy equation

$$\begin{aligned} \frac{\bar{\mathrm{D}}}{\mathrm{D}t}\left(\tfrac{1}{2}\rho \bar{v}_i \bar{v}_i\right) = &-(\bar{v}_i \bar{p})_{,i} + \mu \left[\left(\tfrac{1}{2}\bar{v}_i \bar{v}_i\right)_{,kk} - \bar{v}_{i,k}\bar{v}_{i,k}\right] \\ &- \rho_0 \left[\left(\bar{v}_i \overline{\left(v_i' v_k'\right)}\right)_{,k} - \bar{v}_{i,k}\overline{\left(v_i' v_k'\right)}\right] \\ &- \bar{\rho} g v_z \end{aligned} \tag{11}$$

Turbulent kinetic energy (cont.)

- Now take scalar product of v_i and (9)

$$\rho_0 \left[\frac{1}{2} \frac{\partial}{\partial t} (v_i v_i) + \frac{1}{2} v_k (v_i v_i)_{,k} \right] = -(v_i p)_{,i} + \mu \left[\frac{1}{2} (v_i v_i)_{,kk} - v_{i,k} v_{i,k} \right] - \rho g \delta_{zi} v_i$$

- Subsitute $v_i = \bar{v}_i + v_i'$, $p = \bar{p} + p'$, $v_z = \bar{v}_z + v_z'$, $\rho = \bar{\rho} + \rho'$
- Break the terms up and do some more algebra and take the mean

Turbulent kinetic energy (cont.)

- Mean inertia term:

$$\rho_0 \left[\frac{1}{2} \frac{\partial}{\partial t} \left(\bar{v}_i \bar{v}_i + v_i' v_i' \right) + \frac{1}{2} \left(\bar{v}_x \left(\bar{v}_i \bar{v}_i \right)_{,k} + \bar{v}_k \overline{\left(v_i' v_i' \right)}_{,k} \right. \right.$$

$$\left. \left. + 2 \overline{v_k' \left(v_i' \bar{v}_i \right)} + \overline{v_k' \left(v_i' v_i' \right)} \right) \right]$$

- Mean pressure term: $-\left(\bar{v}_i \bar{p} \right)_{,i} - \left(v_i' p' \right)_{,i}$
- Mean body force term: $-g \bar{\rho} \bar{v}_z - g \overline{\rho' v_z'}$
- Mean viscous term

$$\mu \left[\frac{1}{2} \left(\left(\bar{v}_i \bar{v}_i \right)_{,kk} + \overline{\left(v_i' v_i' \right)}_{,kk} \right) - \left(\bar{v}_{i,k} \bar{v}_{i,k} + \overline{v_{i,k}' v_{i,k}'} \right) \right]$$

Turbulent kinetic energy (cont.)

- Energy of the fluctuation

$$\rho\left[\frac{1}{2}\frac{\partial}{\partial t}\overline{\left(v_i'v_i'\right)}+\bar{v}_k\overline{\left(v_i'v_i'\right)}_{,k}\right]$$
$$=-g\overline{\rho' v_z'}-\overline{\left(v_i'p'\right)}_{,i}+\mu\left[\frac{1}{2}\left(\left(\bar{v}_i\bar{v}_i\right)_{,kk}+\overline{\left(v_i'v_i'\right)}_{,kk}\right)\right.$$
$$\left.-\left(\bar{v}_{i,k}\bar{v}_{i,k}+\overline{v_{i,k}'v_{i,k}'}\right)\right]+\rho_0\left[-\overline{\left(v_i'v_i'\right)}\bar{v}_{i,k}\right.$$
$$\left.-\frac{1}{2}\overline{\left(v_k'v_i'v_i'\right)}_{,k}\right]$$

- The x component of this equation yields the turbulent kinetic energy

$$\frac{\partial \bar{\theta}}{\partial t} = -\frac{\partial}{\partial z}\left(\overline{v_z' \theta'}\right) + \frac{1}{\rho_0 c_v}\frac{\partial q}{\partial z}, \quad q = -k\frac{\partial \theta}{\partial z}$$

$$\frac{\partial \bar{v}_x}{\partial t} = -\frac{\partial}{\partial z}\left(\overline{v_x' v_z'}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\bar{E}'}{2}\right) = -\left(\overline{v_x' v_z'}\right)\frac{\partial \bar{v}_x}{\partial z} - \frac{\partial}{\partial z}\left(\overline{v_z'\left(\frac{p'}{\rho_0} + \frac{E'}{2}\right)}\right) - \frac{g}{\rho_0}\left(\overline{\rho' v_z'}\right) - \varepsilon$$

where

$$E' = v_i' v_i' = (v_x')^2 + (v_z')^2, \qquad \varepsilon = \mu \overline{v_{i,k}' v_{i,k}'}$$

Ordinary differential equations

$$h\frac{\mathrm{d}\theta_s}{\mathrm{d}t} = -\Delta\theta\frac{\mathrm{d}h}{\mathrm{d}t} - \frac{Q_L + H_L + H_S + q_s - q_b(h)}{C_p\rho_0}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(u_s h) = u_*^2$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{E_s h}{2}\right) = \left(\frac{g_s' h}{2} + \frac{u_s'^2}{2}\right)\frac{\mathrm{d}h}{\mathrm{d}t} + C u_*^3 - \left(\frac{\overline{v_3' p'}}{\rho_0} + \frac{\overline{v_3' E'}}{2}\right)\Bigg|_{x_3=H} + \frac{w_*^2}{2} - \epsilon_s h$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{C_F}{C_F + C_E}\left(\frac{C_N^3 u_*^3 + w_*^3}{g_s' h + E_s - C_s u_s^2}\right)$$

Unknown variables: h, u_s, E_s, θ_s

Data collection and model fitting

Coefficients and parameters

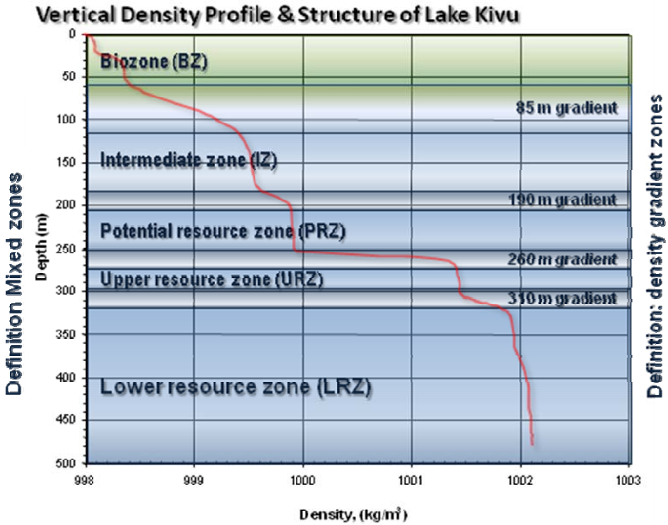
Given

$$C_N = 1.33, \quad C_F = 0.25, \quad C_E = 1.15, \quad C_s = 0.2$$

Calculated

$$H = 485 \text{ km}, \quad L = 90 \text{ km}, \quad W = 50 \text{ km},$$
$$v_1 = 1.9 \text{ m/s}, \quad C_d = 1.5 \times 10^{-}3, \quad q_s = 0.7317,$$
$$C_p = 41747.6 \frac{\text{J}}{\text{K} \cdot \text{kg}}, \quad u_* = 0.0023, \quad \tau = 5.415 \times 10^{-}3,$$
$$g = 9.81 \text{ m/s}^2.$$

Coefficients and parameters (cont.)



Coefficients and parameters (cont.)

Using

$$\rho_0 = 1000\ \frac{\text{kg}}{\text{m}^3}, \quad \rho_s = 998\ \frac{\text{kg}}{\text{m}^3}, \quad \rho_b = 998.3\ \frac{\text{kg}}{\text{m}^3}$$

we obtained

$$\begin{aligned} g_s' &\approx 0.0029, && \text{at} \quad h = 20\ \text{m}, \\ g_s' &\approx 0.0123, && \text{at} \quad h = 80\ \text{m}, \\ N^2 &\approx 0.017, && \text{for} \quad 20\ \text{m} \leq h \leq 80\ \text{m}. \end{aligned}$$

Lake number and Wedderburn number

Wedderburn number

$$W = \frac{g_s' h^2}{u_*^2 L}$$

- h - depth of surface layer
- L - length of lake
- $u_*^2 = \tau_s / \rho_0$ - shear velocity
- τ_s - surface shear due to wind
- $g_s' = Nh^2/2$
- $N^2 = -(g/\rho_0)(\partial \rho / \partial z)$ - square of buoyancy frequency

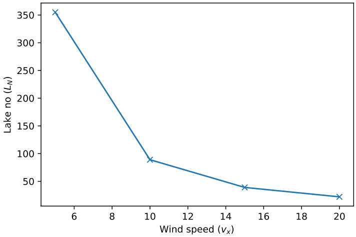
Lake number

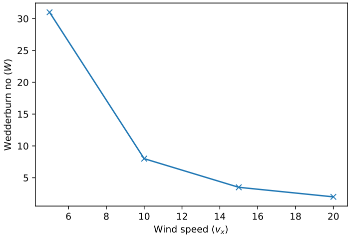
$$L_N = \frac{M_{bc}}{\tau A z_V}$$

- M_{bc} - baryoclinic moment about centre of volume
- τ - surface shear due to wind
- A - area over which τ acts
- z_V - distance from surface to centre of volume

Lake number and Wedderburn number (cont.)

Wind speed (v_1)	5 ms^{-1}	10 ms^{-1}	15 ms^{-1}	20 ms^{-1}
W (surface)	31	8	3.5	2
W (base)	740	185	82	46
L_N	355	89	39	22

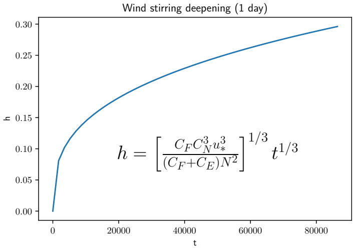


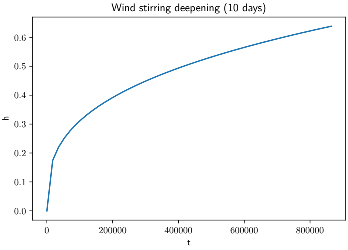


Wind Stirring Deepening (limiting case)

- Assume $w_* = 0$, then
$$g_s' = \frac{Nh^2}{2}$$
- If u_* is constant, then E_s may be constant
$$E_s^{3/2} = \frac{C_N^3 u_*^3}{C_F + C_E}$$
- Obtain
$$C_F E_s^{3/2} = \frac{N^2 h^2}{2} \frac{\mathrm{d}h}{\mathrm{d}t}$$
with solution
$$h = \left[\frac{C_F C_N^3 u_*^3}{(C_F + C_E) N^2} \right]^{1/3} t^{1/3}$$

Wind stirring deepening (cont.)





Conclusion

Conclusion

- From our observations, turbulence is not prevailent in the biozone generally
- However, wind speed during monsoons is fast enough to increase turbulence to potentially influence this layer
- Monsoon winds not fast enough to generate enough turbulence to influence the deeper layers of the lake

Future work

- Non-dimensionalize quantities in equations
- Improve solutions to nonlinear ODEs with Kivu data
- Adapt model with better heat flux equations
- Adapt model to include some type of input from the deep layer

Questions?